Discrete q–derivatives
and
symmetries of q–difference equations

D. Levi [1], J. Negro [2] and M.A. del Olmo [2]

[1]*Dipartimento di Fisica, Universitá Roma Tre and INFN–Sezione di Roma Tre
Via della Vasca Navale 84, 00146 Roma, Italy*

[2]*Departamento de Física Teórica, Universidad de Valladolid,
E-47011, Valladolid, Spain.*

e-mail:levi@fis.uniroma3.it, jnegro@fta.uva.es, olmo@fta.uva.es

February 7, 2008

Abstract

In this paper we extend the *umbral calculus*, developed to deal with difference equations on uniform lattices, to q-difference equations. We show that many of the properties considered for shift invariant difference operators satisfying the umbral calculus can be implemented to the case of the q-difference operators. This *q-umbral calculus* can be used to provide solutions to linear q-difference equations and q-differential delay equations. To illustrate the method, we will apply the obtained results to the construction of symmetry solutions for the q-heat equation and to solve a linear ordinary second order q-difference equation.

1 Introduction

Q-functions appear in many physical problems. They enter in the study of exactly solvable models in statistical mechanics [1], in conformal field theory [2], and are thus very relevant for applications. For example, q-exponential distributions can be obtained following Gibbs' procedure from the stationary conditions on a certain generalized entropy [3]. Standard q–exponential functions are also used to extrapolate between Fermi–Dirac ($q = \infty$) and Bose–Einstein ($q = 0$) statistics, passing through Maxwell–Boltzmann ($q = 1$) statistics [4].

In the case of difference equations one had proved [5] that there exists a very powerful method for systematically discretizing linear differential equations while preserving their properties. Here we extend that method to the case of q-difference equations. We can show that many of the properties considered in [5] for shift invariant difference operators satisfying the umbral calculus [6,7,8,9,10,11] can be extended to the case of the q-difference operators considered in Ref. [12, 13]. For any q-difference operator this q-umbral calculus can be applied to provide solutions to linear q-difference equations and q-differential delay equations. As an illustration, we will apply the method in the construction of symmetry solutions for the q-heat equation and to a linear ordinary q-difference equation.

The paper is organized as follows. In Section 2 we define a q-difference equation in p independent variables and, for the sake of simplicity, just one dependent variable and characterize the symmetry transformations which will leave the equation invariant. Section 3 is devoted to the study of the properties of q-calculus, showing the differences and the similarities between differential and q-difference calculus. In particular in Section 4 we will discuss from an analytic and numerical point of view the simplest q-functions which will be used later, in Section 5, along with a few examples. In Section 5 we present the symmetries of a q-heat equation and solve a differential q-delay equation using the correspondence between q-calculus and differential calculus. Section 6 is dedicated to few conclusive remarks.

2 q-difference equations and its Lie symmetries

Let us consider a linear q-difference equation, involving, for notational simplicity, only one scalar function $u(x)$ of p independent variables $x = (x_1, x_2, \ldots, x_p)$ evaluated at a finite number of points on a lattice. Symbolically we write

$$E_N(x,\ T^a u(x),\ T^{a_{i_1}} \Delta_{x_{i_1}} u(x),\ T^{a_{i_1 i_2}} \Delta_{x_{i_1}} \Delta_{x_{i_2}} u(x), \ldots, \tag{2.1}$$
$$T^{a_{i_1 i_2 \cdots i_N}} \Delta_{x_{i_1}} \Delta_{x_{i_2}} \ldots \Delta_{x_{i_N}} u(x)) = 0, \qquad a = (a_1, a_2, \ldots, a_p),$$

where $a, a_{i-1}, a_{i_1 i_2}, \ldots$ are multi-indices, E_N is some given function of its arguments, $i_1, i_2, \ldots, i_N$ take values between 1 and p. We use the shortening notation $T^a u(x) = \{T^{a_1}_{x_1} T^{a_2}_{x_2} \cdots T^{a_p}_{x_p} u(x)\}$, where a_i, with $i = 1, 2, \ldots, p$, takes values between m_i and n_i, with $m_i,\ n_i$ fixed integers ($m_i \leq n_i$), and the individual q–shift operator $T^{a_i}_{x_i}$ is given by

$$T^{a_i}_{x_i} u(x) = u(x_1, x_2, \ldots, x_{i-1}, q_i^{a_i} x_i, x_{i+1}, \ldots, x_p). \tag{2.2}$$

The other q-shift operators $T^{a_{i_1 i_2}}, \ldots$ are defined in a similar way. The operator Δ_{x_i} is a q_i-difference operator which in the continuous limit when $q_i \to 1$ goes into the partial derivative with respect to the x_i variable (few examples of it are given in Section 3). By q_i we denote a positive parameter which defines a non uniform lattice of the variable x_i $(i = 1, \ldots, p)$.

To study the symmetries of eq.(2.1) we will use the approach introduced in Ref. [14], based on the formalism of evolutionary vector fields for differential equations [15]. As in the case of differential equations the symmetry group of a discrete equation is characterized by those transformations of the equation that carry solutions $u(x)$ into solutions $\tilde{u}(x)$. Moreover we look only for those symmetries which in the continuous limit go over to Lie point symmetries. In such a case the infinitesimal symmetry generators of the symmetry group of equation (2.1) in evolutionary form have the general expression

$$X_e \equiv Q(x, u)\partial_u = \left(\sum_{i=1}^{p} \xi_i(x, T^a u, \{q_j\}_{j=1}^{p}) T^b \Delta_{x_i} u - \phi(x, T^c u, \{q_j\}_{j=1}^{p}) \right) \partial_u, \quad (2.3)$$

with $\xi_i(x, T^a u, \{q_j\}_{j=1}^{p})$ and $\phi(x, T^c u, \{q_j\}_{j=1}^{p})$ such that in the continuous limit go over to $\xi_i(x, u)$ and $\phi(x, u)$, the infinitesimal generators of the corresponding Lie point symmetries. The group transformations are obtained by integrating the differential equation

$$\frac{d\tilde{u}(\tilde{x})}{dg} = Q(\tilde{x}, \tilde{u}(\tilde{x})), \qquad \tilde{u}(\tilde{x}, g = 0) = u(x), \tag{2.4}$$

where g is the group parameter. Eq.(2.4) can be integrated on the characteristics in the continuous limit, however in the discrete case the integration of the corresponding differential q–difference equation can almost never be carried out.

Eq. (2.1) is of order N in the difference operators and of order $i_1 + i_2 + \ldots + i_N$ in the shift operators. Hence, constructing the following prolongation of X_e

$$prX_e = \sum_a T^a Q \partial_{T^a u} + \sum_{i_1} T^{a_{i_1}} Q^{x_{i_1}} \partial_{T^{a_{i_1}} \Delta_{x_{i_1}} u(x)} + \sum_{i_1, i_2} T^{a_{i_1 i_2}} Q^{x_{i_1} x_{i_2}} \partial_{T^{a_{i_1 i_2}} \Delta_{x_{i_1}} \Delta_{x_{i_2}} u(x)} + \ldots$$

$$\tag{2.5}$$

we find that the invariance condition

$$prX_e E_N|_{E_N = 0} = 0 \tag{2.6}$$

must be satisfied. The summations in (2.5) are over all the sites present in (2.1). By $Q^{x_{i_1}}, \quad Q^{x_{i_1} x_{i_2}}, \ldots$ we denote the total variations of Q, i.e.,

$$Q^{x_{i_1}} = \Delta_{x_{i_1}}^T Q, \qquad Q^{x_{i_1} x_{i_2}} = \Delta_{x_{i_1}}^T \Delta_{x_{i_2}}^T Q, \qquad \cdots,$$

where, in the simple case of the q-right derivative Δ_x^+ presented below in eq.(3.1), the partial variation $\Delta_{x_{i_1}}$ is defined by

$$\Delta_{x_{i_1}} f(x_1, \ldots, x_p, u(x_1, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots, x_p), \ldots)$$
$$= \frac{1}{(q_{i_1} - 1)x_{i_1}} [f(x_1, \ldots q_{i_1} x_{i_1}, \ldots, x_p, u(x_1, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots, x_p), \ldots)$$
$$- f(x_1, \ldots, x_p, u(x_1, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots, x_p), \ldots)],$$

3

and the total variation $\Delta^T_{x_{i_1}}$ by

$$\Delta^T_{x_{i_1}} f(x_1, \ldots, x_p, u(x_1, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots, x_p), \ldots) =$$
$$\frac{1}{(q_{i_1}-1)x_{i_1}} [f(x_1, \ldots q_{i_1} x_{i_1}, \ldots, x_p, u(x_1, \ldots q_{i_1} x_{i_1}, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots q_{i_1} x_{i_1}, \ldots, x_p), \ldots)$$
$$- f(x_1, \ldots, x_p, u(x_1, \ldots, x_p), \Delta_{x_j} u(x_1, \ldots, x_p), \ldots)].$$

Notice that expressions (2.3–2.5) are analogous to those of the continuous case [15] and can be derived in a similar way [14].

The group generated by the prolongations also transforms solutions into solutions, and $\Delta_{x_i} u, \Delta_{x_i} \Delta_{x_j} u, \ldots$ (up to order N) into the variations of $\tilde{u}$ with respect to the corresponding $\tilde{x}_i$.

The symmetries of equation (2.1) are given by condition (2.6), which give rise to a set of determining equations for ξ_i and ϕ obtained as coefficients of the linearly independent expressions in the discrete derivatives $\Delta_{x_i} u$, $\Delta_{x_i x_j} u,)$.

The Lie commutators of the vector fields X_e are obtained by commuting their first prolongations and projecting them onto the symmetry algebra $\mathcal{G}$, i.e.,

$$[X_{e_1}, X_{e_2}] = [pr^1 X_{e_1}, pr^1 X_{e_2}]|_{\mathcal{G}} \tag{2.7}$$
$$= \left(Q_1 \frac{\partial Q_2}{\partial u} - Q_2 \frac{\partial Q_1}{\partial u} + Q_1^{x_i} \frac{\partial Q_2}{\partial u_{x_i}} - Q_2^{x_i} \frac{\partial Q_1}{\partial u_{x_i}} \right) \partial_u,$$

where the ∂u_{x_i} terms disappear after projection onto $\mathcal{G}$.

The formalism presented above may become quite involved, but the situation is simpler for linear equations where we can use a reduced Ansatz. In this case we can assume that the evolutionary vectors (2.3) have the form

$$X_e = \left(\sum_i \xi_i(x, T^a, q_j) \Delta_{x_i} u - \phi(x, T^a, q_j) u \right) \partial_u. \tag{2.8}$$

The vector fields X_e can be written as $X_e = (\hat{X} u) \partial_u$ with

$$\hat{X} = \sum_i \xi_i(x, T^a, q_j) \Delta_{x_i} - \phi(x, T^a, q_j). \tag{2.9}$$

Notice that $\hat{X}$ span a subalgebra of the whole Lie symmetry algebra (see Ref. [14]).

If the system is nonlinear the simplification (2.8) is too restrictive and is almost impossible to get a non trivial result since the number of terms to consider is a priori infinite.

3 q–Calculus

In this section we present the generalities of q–calculus [16]. We will restrict ourself for the sake of simplicity to one independent variable. Moreover, in the following we will consider

just the simplest q–derivatives (at the right, at the left and symmetric, respectively)

$$\Delta_x^+ = \frac{1}{q_x^+ x}(T_x - 1), \tag{3.1}$$

$$\Delta_x^- = \frac{1}{q_x^- x}(1 - T_x^{-1}), \tag{3.2}$$

$$\Delta_x^{\rm s} = \frac{1}{q_x^{\rm s} x}(T_x - T_x^{-1}), \tag{3.3}$$

where q_x is a real dilation positive parameter associated to the variable x, $q_x^i, i = \pm, s$ are given by

$$q_x^+ = q_x - 1 \qquad q_x^- = 1 - \frac{1}{q_x} \qquad q_x^{\rm s} = q_x - \frac{1}{q_x}, \tag{3.4}$$

and T_x is a q–dilation

$$T_x f(x) = f(q_x x) \qquad T_x^{-1} f(x) = f(x/q_x). \tag{3.5}$$

The operator T_x is the one-dimensional reduction of eq. (2.2). Formally we have

$$T_x = e^{(q_x - 1)x \partial_x}. \tag{3.6}$$

When we do not specify which q–derivative we are using we will write just Δ_x.

It is easy to see that, due to the form of the q–derivative considered (3.1 - 3.3), the shift operator and the q–derivatives do not commute. So the q–derivative operators are not shift invariant operators and do not satisfy one of the basic conditions in the umbral calculus [6, 7, 8, 9]. We can, however, carry out an umbral calculus even if our q–derivative are not shift invariant that we will call q–*umbral calculus*. The q–umbral calculus is defined in the same way as the standard umbral calculus, but we do not require the commutativity of the delta operators with the shift operators. The absence of this property has no consequence in all the results presented in the following.

We can easily find that

$$[\Delta_x, x] = \begin{cases} T_x & \text{for} & \Delta_x^+ \\ T_x^{-1} & \text{for} & \Delta_x^- \\ \frac{1}{1+q_x}(q_x T_x + T_x^{-1}) & \text{for} & \Delta_x^{\rm s} \end{cases} \tag{3.7}$$

If instead of the standard commutator, we consider the q–commutator defined as $[A, B]_{q^+} = AB - qAB$ and $[A, B]_{q^-} = AB - (1/q)AB$, we have

$$[\Delta_x, x]_q = 1. \tag{3.8}$$

The result (3.8) is not valid in the case of the symmetric q–derivative. Moreover, the expression (3.8) does not satisfy the Leibniz rule and, thus, the expression $[\Delta_x, x^n]_q$ becomes

5

more and more complicate when we consider $n = 2, 3, \cdots$. So, here in the following we will consider just the standard commutator.

We can define an operator β_x, depending on T_x, in the spirit of umbral calculus [9,10], such that

$$[\beta_x, T_x] = 0, \qquad [\Delta_x, \beta_x x] = 1. \tag{3.9}$$

For the three q–derivatives introduced above (3.1) we can find the following explicit expressions of β_x:

$$\beta_x^+ = (q_x - 1)x\partial_x(T_x - 1)^{-1} = q_x^+ x\partial_x(T_x - 1)^{-1},$$

$$\beta_x^- = (1 - \frac{1}{q_x})x\partial_x(1 - T_x^{-1})^{-1} = q_x^- x\partial_x(1 - T_x^{-1})^{-1}, \tag{3.10}$$

$$\beta_x^s = (q_x - \frac{1}{q_x})x\partial_x(T_x - T_x^{-1})^{-1} = q_x^s x\partial_x(T_x - T_x^{-1})^{-1}.$$

These may not be the only possible definitions.

It is easy to prove that always

$$\beta_x x \Delta_x = x\partial_x. \tag{3.11}$$

Moreover, due to the presence of the ∂_x operator in the definition of β_x, the q–umbral correspondence of an explicit x–dependent differential equation will give rise to a q–differential delay equation [17].

We can reexpress the functions β_x as an infinite series in terms of the shift operators, thus proving that they commute with the shift operators. From (3.6) we get

$$x\partial_x = \frac{1}{q_x - 1}\ln T_x = \frac{1}{q_x - 1}\ln(1 + (T_x - 1)) = \frac{1}{q_x - 1}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(T_x - 1)^n}{n}. \tag{3.12}$$

Consequently

$$\beta_x^+ = \frac{q_x^+}{q_x - 1}\sum_{n=0}^{\infty}(-1)^n\frac{(T_x - 1)^n}{n + 1}. \tag{3.13}$$

Similarly from $T_x^{-1} = \exp\left(-(q_x - 1)x\partial_x\right)$

$$x\partial_x = -\frac{1}{q_x - 1}\ln T_x^{-1} = -\frac{1}{q_x - 1}\ln(1 + (T_x^{-1} - 1)) = \frac{1}{q_x - 1}\sum_{n=1}^{\infty}(-1)^n\frac{(T_x^{-1} - 1)^n}{n} \tag{3.14}$$

we get

$$\beta_x^- = \frac{q_x^-}{q_x - 1}\sum_{n=0}^{\infty}(-1)^n\frac{(T_x^{-1} - 1)^n}{n + 1} = \frac{q_x^-}{q_x - 1}\sum_{n=0}^{\infty}\frac{(1 - T_x^{-1})^n}{n + 1}. \tag{3.15}$$

Finally, for the symmetric derivative, as

$$\frac{T_x - T_x^{-1}}{2} = \sinh((q_x - 1)x\partial_x) \tag{3.16}$$

6

we find that

$$x\partial_x = \frac{1}{q_x - 1}\sinh^{-1}\frac{T_x - T_x^{-1}}{2} = \frac{1}{q_x - 1}\sum_{n=1}^{\infty}(-1)^{n+1}C_n\frac{(T_x - T_x^{-1})^{2n-1}}{2^{2n-1}}, \qquad (3.17)$$

where the coefficients C_n are given by

$$C_1 = 1, \qquad C_n = \frac{\prod_{k=2}^{n}(2k-3)}{(2n-1)\prod_{k=2}^{n}(2k-2)}, \qquad \forall n \geq 2. \qquad (3.18)$$

So, we have:

$$\beta_x^{\rm s} = \frac{q_x^{\rm s}}{\ln q_x}\sum_{n=0}^{\infty}(-1)^n C_{n+1}\frac{(T_x - T_x^{-1})^{2n}}{2^{2n+1}}. \qquad (3.19)$$

For all functions f and g, entire in $\beta_x x$, the Leibniz rule takes the form

$$[\Delta_x, fg] = [\Delta_x, f]g + f[\Delta_x, g]. \qquad (3.20)$$

Note that the expression of Leibniz's rule (3.20) is exactly the same as for discrete derivatives [5]. For the sake of brevity we will write $D_x f := [\Delta_x, f]$. The expression (3.20) is operatorial. If we want to have a functional expression we need to project it by acting on a constant function as 1. In this case, however, the Leibniz rule (3.20) is an identity.

Taking into account the Leibniz rule (3.20) we can prove for the three q–derivatives introduced above the following property

$$[\Delta_x, (\beta_x x)^n] = D_x(\beta_x x)^n = n(\beta_x x)^{n-1}, \qquad \forall n \in \mathbb{N}, \qquad (3.21)$$

thus showing that $(\beta_x x)^n$ are basic polynomials for the operator D_x, and, when projected, for Δ_x. So, we have defined an operator D_x which on functions of $\beta_x x$ have the same properties as the normal derivatives ∂_x on functions of x. Hence, we can say that whatsoever is valid for differential equations can be also valid for the D_x operators provided we consider instead of functions of x functions depending on $\beta_x x$. This is the content of the q–umbral correspondence. In the case of linear differential equations, when the derivations act linearly on functions, the projection procedure will transform the operator D_x into Δ_x and we will get q–difference equations.

Let us analyze the meaning of the basic polynomial operators $(\beta_x x)^n$ for the three q–derivatives operators. Since

$$(T_x - 1)x = x(q_x T_x - 1), \qquad\qquad (T_x - 1)^{-1}x = x(q_x T_x - 1)^{-1}, \qquad (3.22)$$

$$(1 - T_x^{-1})x = x(1 - \frac{1}{q_x}T_x^{-1}), \qquad\qquad (1 - T_x^{-1})^{-1}x = x(1 - \frac{1}{q_x}T_x^{-1})^{-1}, \qquad (3.23)$$

$$(T_x - T_x^{-1})x = x(q_x T_x - \frac{1}{q_x}T_x^{-1}1), \qquad (T_x - T_x^{-1})^{-1}x = x(q_x T_x - \frac{1}{q_x}T_x^{-1})^{-1}, \qquad (3.24)$$

we get

$$\beta_x^+ x = q_x^+ x(1 + x\partial_x)(q_x T_x - 1)^{-1},$$

7

$$\beta_x^- x = q_x^- x (1 + x\partial_x)(1 - \frac{1}{q_x}T_x^{-1})^{-1}, \tag{3.25}$$

$$\beta_x^s x = q_x^s x (1 + x\partial_x)(q_x T_x - \frac{1}{q_x}T_x^{-1})^{-1}. $$

These expressions have an operational character. In order to reduce them to a function we have to project them by acting $\beta_x x$ on a constant. So, taking into account that $T_x\, 1 = 1$ and $T_x^{-1}\, 1 = 1$, in all the three cases we have

$$\beta_x x\, 1 = x. \tag{3.26}$$

One can demonstrate by induction that

$$(\beta_x x)^n\, 1 = (\beta_x x)(\beta_x x)\cdots(\beta_x x)\, 1 = \frac{n!}{[n]_q!}x^n, \qquad \forall n \in \mathbb{N}^+, \tag{3.27}$$

where

$$[n]_q^+ = \frac{q^n - 1}{q - 1}, \qquad [n]_q^- = \frac{1 - q^{-n}}{1 - q^{-1}}, \tag{3.28}$$

$$[n]_q^s = \frac{q}{q^2 - 1}\frac{q^{2n} - 1}{q^n} = \frac{1}{q^{n-1}}\sum_{k=1}^{n} q^{2k-2}, \tag{3.29}$$

and

$$[n]_q! = [n]_q[n-1]_q\cdots[1]_q. \tag{3.30}$$

Consequently, if we consider an entire function like the exponential, we have

$$e^{\lambda\beta_x x}\, 1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(\beta_x x)^n 1 = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{[n]_q!}. \tag{3.31}$$

Therefore, by the q–umbral correspondence, the exponential function becomes

$$e^{\lambda x} = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{n!} \longrightarrow e^{\lambda\beta_x x}1 = \sum_{n=0}^{\infty} \lambda^n \frac{x^n}{[n]_q!}. \tag{3.32}$$

Let us consider the gaussian function, that we will be using later in this work. It takes the form

$$e^{-\lambda(\beta_x x)^2}\, 1 = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}(\beta_x x)^{2n}1 = \sum_{n=0}^{\infty} (-\lambda)^n \frac{(2n)!}{n!}\frac{x^{2n}}{[2n]_q!}. \tag{3.33}$$

By considering an arbitrary point x_0 and an arbitrary real constant a we have

$$D_x(\beta_x x + x_0)^a = a(\beta_x x + x_0)^{a-1}. \tag{3.34}$$

The proof of eq. (3.34) is based on the idea that the differential equation

$$(x + x_0)\partial_x f = af, \tag{3.35}$$

whose solution is $f = (x + x_0)^a$, can be transformed by the q–umbral correspondence into the discrete equation

$$(\beta_x x + x_0) D_x (\beta_x x + x_0)^a = a(\beta_x x + x_0)^a, \tag{3.36}$$

which has formally the same solution in power series (substituting x by $\beta_x x$ in the discrete case). Effectively,

$$(x + x_0)^a = \sum_{n=0}^{\infty} \frac{x_0^{a-n} \prod_{k=0}^{n-1}(a-k)}{n!} x^n \tag{3.37}$$

while

$$(\beta_x x + x_0)^a = \sum_{n=0}^{\infty} \frac{x_0^{a-n} \prod_{k=0}^{n-1}(a-k)}{n!} (\beta_x x)^n. \tag{3.38}$$

After projection we get

$$(\beta_x x + x_0)^a \, 1 = \sum_{n=0}^{\infty} \frac{x_0^{a-n} \prod_{k=0}^{n-1}(a-k)}{n!} (\beta_x x)^n \, 1 = \sum_{n=0}^{\infty} \frac{x_0^{a-n} \prod_{k=0}^{n-1}(a-k)}{[n]_q!} x^n. \tag{3.39}$$

$\square$

4 q–Umbral functions

In this section we consider some basic discrete functions obtained by means of the q-umbral method and discuss their range of validity taking as reference the corresponding solution of the difference equation and the continuous corresponding function. As concrete examples we shall consider as q–derivative operator two of the q–derivatives of the previous sections: the right Δ^+ and the symmetric Δ^s. We present here two of the functions which appear in the symmetry reduction of the heat equation, the exponential and gaussian functions (3.32), (3.33), which exemplify the kind of results one can obtain from the q–umbral calculus for entire functions. These functions are described, respectively, by the ordinary differential equations:

$$\frac{df_e(x)}{dx} = \lambda f_e(x) \tag{4.1}$$

and

$$\frac{df_g(x)}{dx} = -2\lambda x f_g(x). \tag{4.2}$$

Eq.(4.1) has as solution the function $f_e(x) = ae^{\lambda x}$ and eq.(4.2) has as solution the function $f_g(x) = be^{-\lambda x^2}$, where a and b are arbitrary integration constants.

4.1 q–Exponential functions

From the q-umbral correspondence the difference equation satisfied by the q–exponential is (see eq.(4.1))

$$\Delta E_q(x) = \lambda E_q(x). \tag{4.3}$$

We will discuss always the domain $x > 0$ for all real λ. The negative values $x < 0$ can be obtained by changing the sign of λ.

The right exponential

From (3.1) the difference equation (4.3) becomes

$$E_q(qx) = [1 + (q-1)\lambda x]E_q(x). \tag{4.4}$$

The solution can be expressed as a product

$$E_q(q^n x_0) = \prod_{j=0}^{n-1} \left[1 + (q-1)\lambda q^j x_0\right] E_q(x_0), \quad n \in \mathbb{N}^+, \tag{4.5}$$

$$E_q(q^{-n} x_0) = \prod_{j=1}^{n} \frac{1}{[1 + (q-1)\lambda q^{-j} x_0]} E_q(x_0), \quad n \in \mathbb{N}^+. \tag{4.6}$$

We have four different behaviours of the q–exponential function according to the values of q and λ, namely $q \gtrless 1$ and $\lambda \lessgtr 0$.

- $q > 1$

    - Let us at first consider the case $\lambda > 0$. The recurrence of the difference equation implies that $E_q(x)$ is an (monotonous) increasing function in x.

      By the q–umbral correspondence the solution of the equation (4.1) can be also obtained by q–umbralizing the series representation of the exponential function. In such a case we have

      $$\widetilde{E}_q(x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k (q-1)^k}{\prod_{j=1}^{k}(q^j - 1)}. \tag{4.7}$$

      This solution converges for all $x > 0$, so that it gives the unique solution (4.5-4.6) of (4.4) in all the domain.

    - In the case $\lambda < 0$ we see that the recurrence leads to a decreasing function as long as $1 - (q-1)\lambda x > 0$. However, for further values of x, where $1 - (q-1)\lambda x < 0$, the function oscillates with higher diverging amplitudes. For the particular point $x_0 = 1/((q-1)|\lambda|)$ we have $E_q(qx_0) = 0$, therefore also $E_q(q^n x_0) = 0, \forall n \in \mathbb{N}$. This means that in the q–lattice $x_n = x_0 q^n, n \in \mathbb{Z}$, the q-exponential decreases for n negative, i.e. for values of x less than x_0 and vanishes after, for n positive, avoiding the oscillations.

As $q \to 1$ the point $x_0 \to \infty$, so that the q–exponential in that lattice becomes closer and closer to the (continuous) exponential. The q–umbral function (4.7) displays these same features as it is shown in Fig. 1.
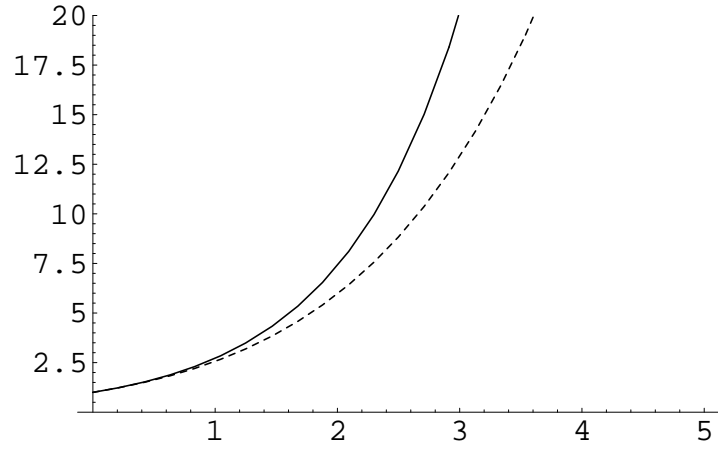
10

Fig. 1A. Plotting of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = 1$ and $q = 1.3$.
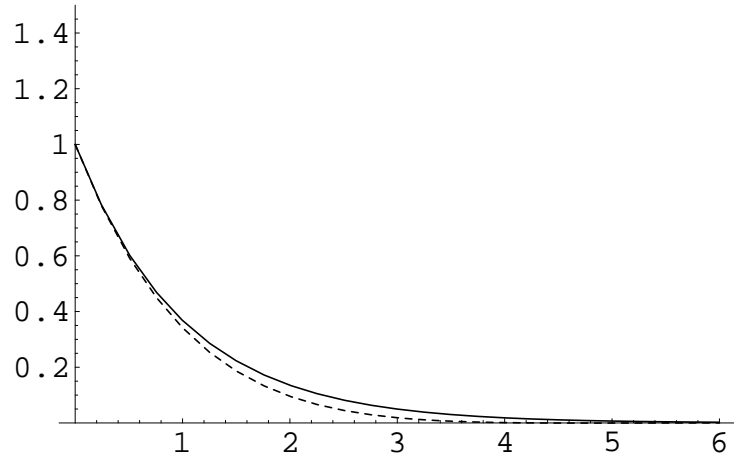

Fig. 1B. Plotting of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = -1$ and $q = 1.3$.
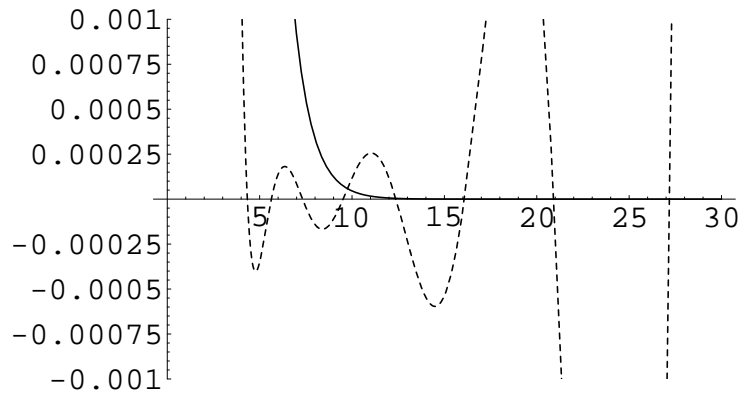
Fig. 1C. Enlarged plotting of the exponential function (continuous line) and right q-exponential function (dashed line) for $\lambda = -1$ and $q = 1.3$.

- $q < 1$

  - For $\lambda > 0$ and for big values of n one can always find $[1+(q-1)\lambda q^n x_0] > 0$ and the solution is a monotonous increasing function for small values of x_0. For the value x_0 such that $[1+(q-1)\lambda x_0] = 0$, the solution is not defined, in fact it diverges in that point, and, consequently, also diverges in the lattice $x_0 q^{-n}$, $n \in \mathbb{N}$. For the rest of the points $x = x_0 q^j$ such that $[1+(q-1)\lambda q^j x_0] < 0$, that is, for the values $x > x_0$, the solution oscillates, changing the sign alternatively, with divergences. The amplitude of the oscillations tends to zero, as $x \to +\infty$.

  - For $\lambda < 0$ the solution is a monotonous decreasing function that in the limit $x \to 0$ goes to zero.

    For the q–umbral series the radius of convergence is given by $R = (|q-1||\lambda|)^{-1}$. The q–umbral function $\widetilde{E}_q(x)$ has two vertical asintotes at the symmetric points $x = \pm R$. Therefore, inside the range of convergence (where there are no oscillations) the q–umbral function reproduces correctly the solution $E_q(x)$, but for $x > |R|$ where it does not converges it supplies no information about the solution. These features are illustrated in Fig. 2.
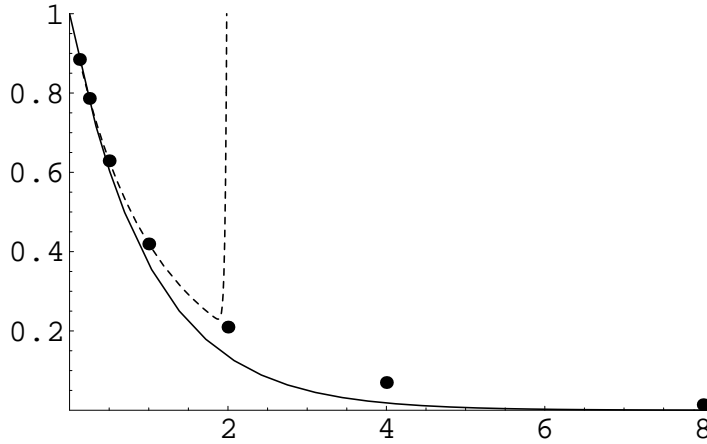

Fig. 2A. Plotting of the exponential function (continuous line) and right q-exponential function (dashed line) and the solution of eq. (4.4) given by the points, for $\lambda = -1$ and $q = 0.5$.
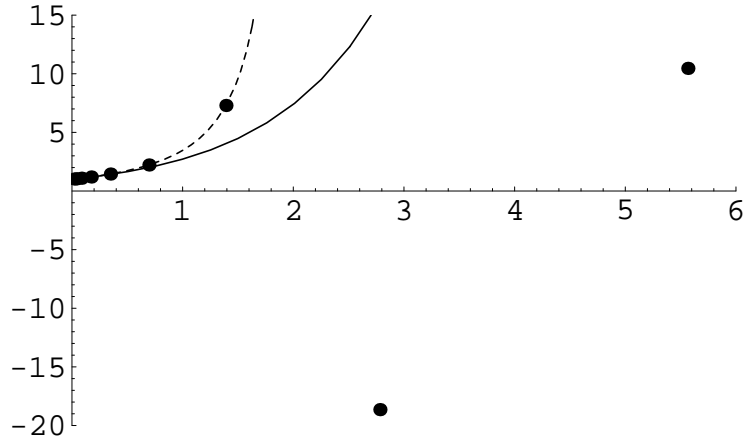
Fig. 2B. Plotting of the exponential function (continuous line) and right q-exponential function (dashed line) and the solution of eq. (4.4) given by the points, for $\lambda = 1$ and $q = 0.5$.
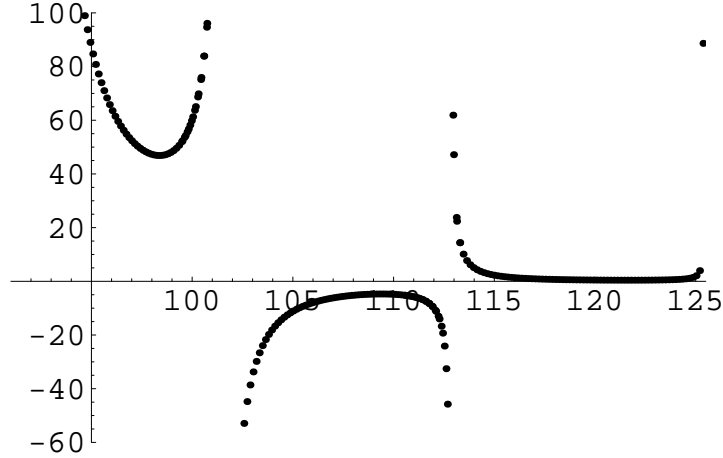

Fig. 2C. Enlarged plotting of the solution of eq. (4.4), for $\lambda = 1$ and $q = 0.5$.

The symmetric exponential

When we consider the Δ^s operator (3.3), the difference equation (4.3) becomes a three term relation given by

$$E_q^s(qx) = (q - 1/q)\lambda x E_q^s(x) + E_q^s(x/q). \tag{4.8}$$

So, there are two independent solutions for the symmetric q-exponential. It is not necessary to distinguish the cases $q > 1$ and $q < 1$ because they play a symmetric role. For $\lambda > 0$ relation (4.8) gives growing functions in the x variable. For $\lambda < 0$ the function initially is decreasing but after a certain point the discrete solutions start to oscillate.

The q–umbral solution to the recurrence equation (4.8) is given by

$$\widetilde{E}_q^s(qx) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k (q - 1/q)^k}{\prod_{j=1}^{k}(q^j - 1/q^j)}. \tag{4.9}$$

13

This series converges for all x, so that it provides one of the solutions to the recurrence (4.8), and in the limit $q \to 1$ it goes into the continuous exponential. The behaviour of the umbral solution $\widetilde{E}_q^s$ presents the features of the recurrence above described, i. e., it approaches the continuous exponential up to the first zero (for $\lambda < 0$) but it becomes wildly oscillating beyond that point (see Fig. 3).
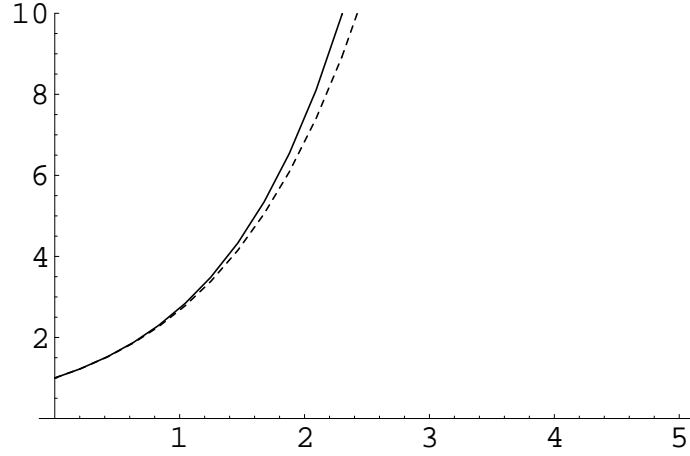

Fig. 3A. Plotting of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = 1$ and $q = 1.3$.
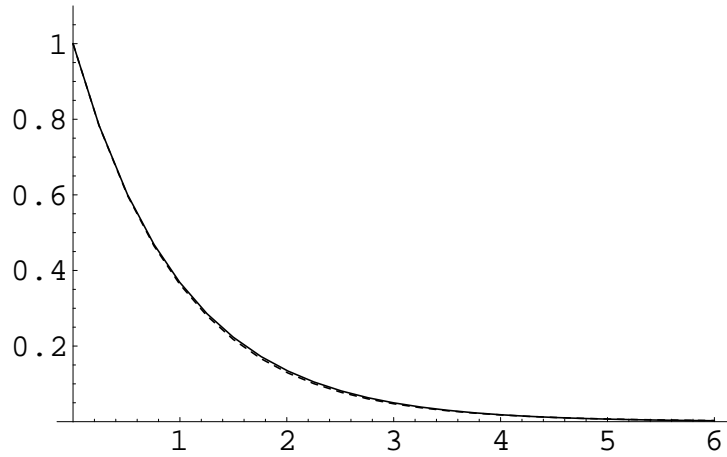
Fig. 3B. Plotting of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = -1$ and $q = 1.3$.
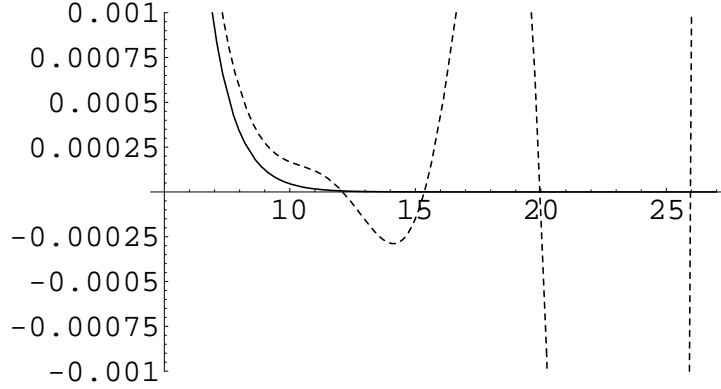
Fig. 3C. Enlarged plotting of the exponential function (continuous line) and symmetric q-exponential function (dashed line), for $\lambda = -1$ and $q = 1.3$.

Comparing Fig. 1 and Fig. 3 we can see that the symmetric exponential function gives a better approximation than the right exponential. However, it can be easily shown that there is no initial condition for equation (4.8) such that the symmetric q–exponential vanishes for all subsequent points, as it was the case for the right exponential. So, there is no way to avoid the oscillations.

4.2 q-Gaussians

In this case eq.(4.2), which has as a solution the gaussian function, becomes the q–difference equation

$$\Delta G_{q,\lambda}(x) = -\lambda x \beta_x G_{q,\lambda}(x). \tag{4.10}$$

In the following we will discuss briefly the cases of the right and symmetric q–gaussians.

The right gaussian

The difference equation (4.10) becomes a three–term recurrence equation

$$q^{-1}G_{q,\lambda}^+(q^2 x) - (q^{-1} + 1)G_{q,\lambda}^+(qx) + G_{q,\lambda}^+(x) = -\lambda(q-1)^2 x(x^2 \partial_x + 1)G_{q,\lambda}^+(x). \tag{4.11}$$

However, now eq. (4.11) is a differential–difference equation. So, it is quite difficult to find directly the solutions or even to discuss the general behaviour of its solutions. We will assume, as shown in the case of the exponential function, that, whenever the q–umbral series is convergent it will converge to the solution of the difference equation.

The q–umbral series supplies a solution to eq. (4.11), given by

$$\widetilde{G}_{q,\lambda}^+(x) = \sum_{k=0}^{\infty} \frac{(-\lambda x^2)^k 2^k (2k-1)!(q-1)^{2k}}{\prod_{j=1}^{2k}(q^j - 1)}. \tag{4.12}$$

For $q > 1$ the series converges for all x, but for $q < 1$ the series diverges everywhere (for $x \neq 0$). Some plottings of $\widetilde{G}_{q,\lambda}^+$ are shown in Fig. 4. The behaviour is similar to that of the q–exponential. Whenever we have a decreasing function of x, at a certain point

15

it vanishes and beyond that point it starts to oscillate with increasing amplitudes, thus departing from the behaviour of the continuous gaussian function.
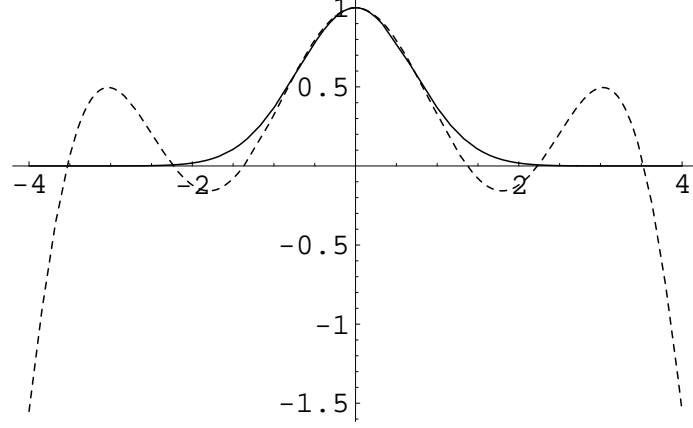

Fig. 4A. Plotting of the gaussian function (continuous line) and right q-gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.
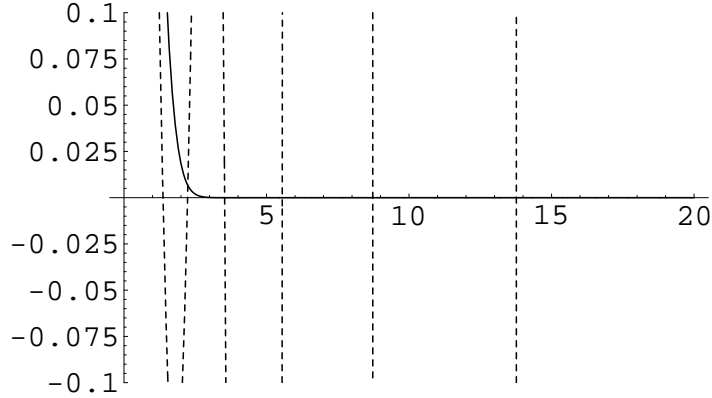

Fig. 4B. Enlarged plotting of the gaussian function (continuous line) and right q-gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.

The symmetric gaussian

The recurrence relation of eq. (4.10) for this case is an even more involved differential difference equation than eq.(4.11), so we prefer not to write it down. The q–umbral series solution is given by

$$\widetilde{G}^s_{q,\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-\lambda x^2)^k (2k)! (q - 1/q)^{2k}}{k! \prod_{j=1}^{2k}(q^j - 1/q^j)}. \tag{4.13}$$

The radius of convergence is $R = \infty$. For the symmetric gaussian function we have similar results as for the right gaussian, which, however, are valid for any value of q (see Fig.5).
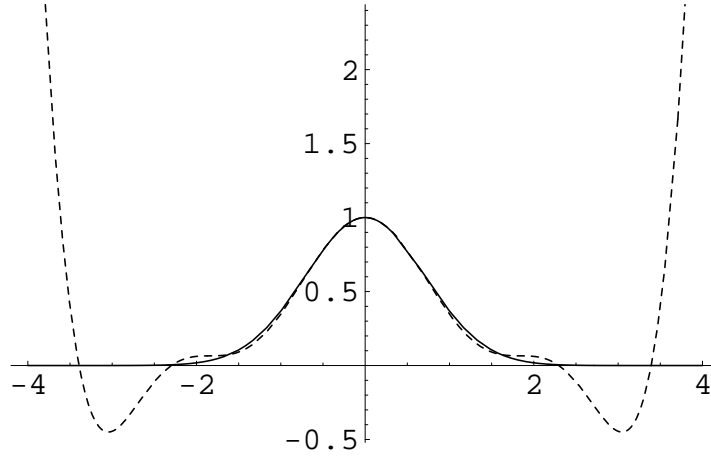
16

Fig. 5A. Plotting of the gaussian function (continuous line) and symmetric q-gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.
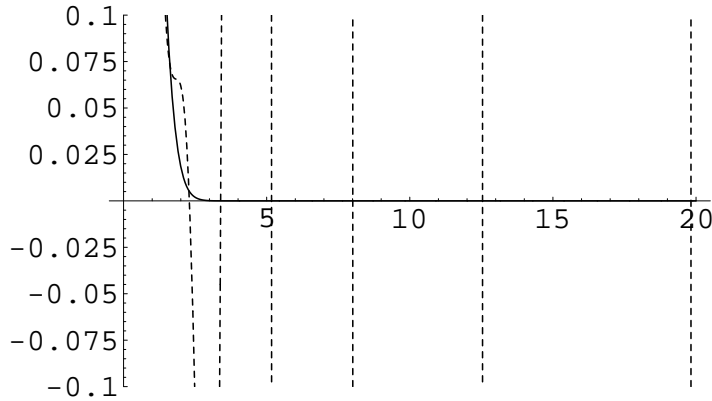

Fig. 5B. Enlarged plotting of the gaussian function (continuous line) and symmetric q-gaussian function (dashed line), for $\lambda = 1$ and $q = 1.3$.

In conclusion, we can say that the decreasing asymptotic behaviour of the classical functions is not fully reproduced by the corresponding umbral q–functions. In that region the discrete functions approach the classical ones up to a point where they vanish, but beyond this point they oscillate going far away from the continuous analogues. Therefore, a good parameter measuring the radius of the domain where the q–functions imitate the continuous functions is given by the first zero in the region of asymptotic behaviour. This is depicted in Fig. 6 for the q–exponentials and the q–gaussians.

From these plottings we evince that the domain of convergence of the q–exponential function to the exponential function increases in a monotonic continuous way as $q \to 1$ for the right exponential while in the symmetric case there are discontinuities for small values of q. The situation is slightly different in the case of the gaussian function, when, as q decreases also the domain decreases up to a minimum λ dependent value q_0. Below q_0 the domain increases as $q \to 1$. In the case of the symmetric gaussian function we again have that for q small the function is discontinuous.
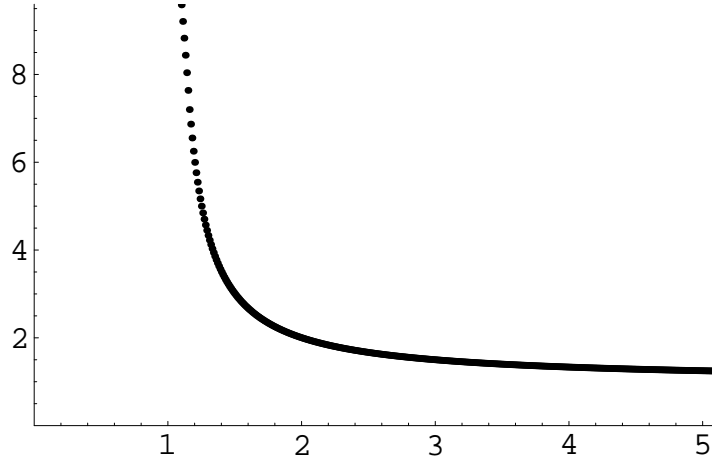
17

Fig. 6A. Plotting of the position of first zero of the right q-exponential as function of q for $\lambda = -1$
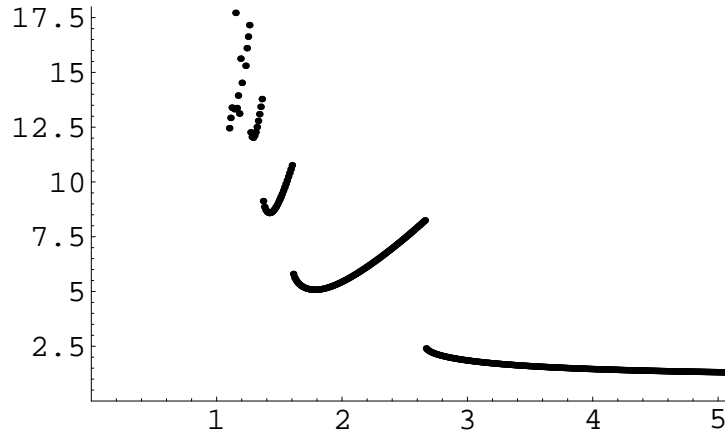

Fig. 6B. Plotting of the position of first zero of the symmetric q-exponential as function of q for $\lambda = -1$
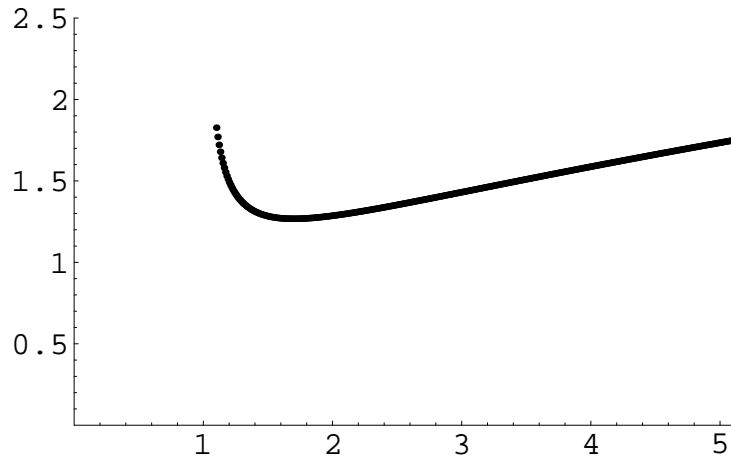

Fig. 6C. Plotting of the position of first zero of the right q-gaussian as function of q for $\lambda = 1$
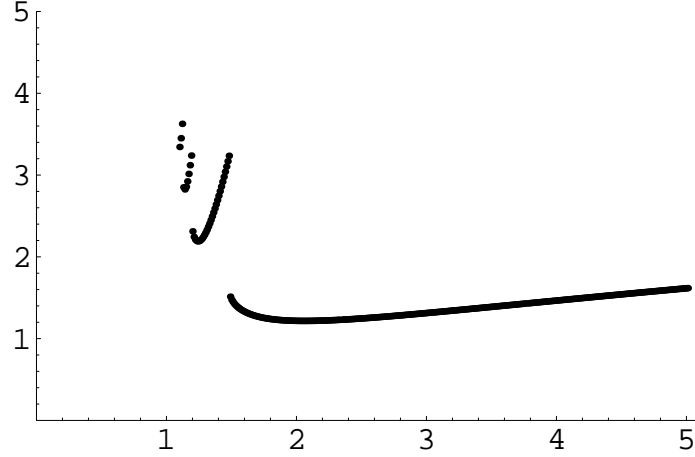
Fig. 6D. Plotting of the position of first zero of the symmetric q-gaussian as function of q for $\lambda = 1$

5   Examples

5.1   Discrete heat q–equation

Taking into account the q-umbral correspondence

$$\partial_x \to \Delta_x, \qquad x \to \beta_x x \tag{5.1}$$

we obtain from any linear differential equation with constant coefficients an operator equation which, when projected, gives us a q–discrete equation. In the case of the heat equation we get

$$(\partial_t - \partial_{xx}^2)u = 0 \quad \implies \quad (\Delta_t - \Delta_{xx})u = 0 \tag{5.2}$$

Let us consider now the problem of obtaining the symmetries for the q–discrete eq.(5.2). We can apply the q–umbral correspondence also to the determining equations, as they are linear in the infinitesimal coefficients ξ, τ and f. So, making use of the Leibniz rule we obtain the following set of determining equations:

$$D_x(\tau) = 0, \tag{5.3}$$

$$D_t(\tau) - 2Q^x(\xi) = 0, \tag{5.4}$$

$$D_t(\xi) - Q^{xx}(\xi) - 2Q^x(f) = 0, \tag{5.5}$$

$$D_t(f) - Q^{xx}(f) = 0, \tag{5.6}$$

where $D_x(\tau)1 = [\Delta_x, \tau]1 = \Delta_x\tau$. Moreover

$$D_{xx}(f)1 = D_x(D_x(f))1 = [\Delta_x, [\Delta_x, f]]1 = \Delta_x\Delta_x f.$$

Note that these determining equations have formally the same expression for all q–derivatives, for the continuous derivatives and in the discrete case studied in [5]. From

19

eq.(5.1), the solution of this system (5.3) is

$$\tau = \tau_2(\beta_t t)^2 + \tau_1(\beta_t t) + \tau_0, \tag{5.7}$$

$$\xi = \frac{1}{2}(\tau_1 + 2\tau_2(\beta_t t))(\beta_x x) + \xi_1(\beta_t t) + \xi_0, \tag{5.8}$$

$$f = \frac{1}{4}\tau_2(\beta_x x)^2 + \frac{1}{2}\tau_2(\beta_t t) + \frac{1}{2}\xi_1(\beta_x x) + \gamma, \tag{5.9}$$

where τ_0, τ_1, τ_2, ξ_0, ξ_1 and γ are arbitrary functions of T_x, T_t and of q_x and q_t.

By a suitable choice of the functions τ_1, τ_2, τ_0, ξ_1, ξ_0 and γ we get the following representation of the symmetries:

$$P_0^q = (\Delta_t u)\partial_u, \tag{5.10}$$

$$P_1^q = (\Delta_x u)\partial_u, \tag{5.11}$$

$$W^q = u\partial_u, \tag{5.12}$$

$$B^q = (2(\beta_t t)\Delta_t u + (\beta_x x)\Delta_x u)\partial_u, \tag{5.13}$$

$$D^q = (2(\beta_t t)\Delta_t u + (\beta_x x)\Delta_x u + \frac{1}{2}u)\partial_u, \tag{5.14}$$

$$K^q = ((\beta_t t)^2\Delta_t u + (\beta_t t)(\beta_x x)\Delta_x u + \frac{1}{4}(\beta_x x)^2 u + \frac{1}{2}(\beta_t t)u)\partial_u, \tag{5.15}$$

that close into a 6–dimensional Lie algebra, isomorphic to the symmetry algebra of the continuous heat equation. Another realization of this algebra was obtained in [19], by a different procedure and used to find symmetric solutions of the discrete heat equation.

Now we use these symmetries to construct few solutions of the q–discrete equation (5.2). Taking into account the symmetries P_0^q and P_1^q, choosing the q–parameters in such a way that $q_x = q_t = q$ and, using the variable separation method [20], we can write the solution of the q–heat equations as

$$u(\beta_t t, \beta_x x) = v(\beta_t t)w(\beta_x x). \tag{5.16}$$

Then eq.(5.2) reads

$$(\Delta_t v(\beta_t t))w(\beta_x x) - v(\beta_t t)(\Delta_{xx})w(\beta_x x)) = 0. \tag{5.17}$$

From eq.(5.17) we deduce that the functions v and w must satisfy the following equations

$$\Delta_t v(\beta_t t) = \lambda v(\beta_t t), \qquad \Delta_x w(\beta_x x) = \sqrt{\lambda}w(\beta_x x). \tag{5.18}$$

By means of eq.(4.3), we have

$$v(t) = v(\beta_t t)1 = e^{\lambda \beta_t t}1 = \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{[n]_q!} = e_q^{\lambda t} \tag{5.19}$$

and

$$w(x) = w(\beta_x x)1 = e^{\sqrt{\lambda}\beta_x x}1 = \sum_{n=0}^{\infty} \frac{\lambda^{n/2}x^n}{[n]_q!} = e_q^{\sqrt{\lambda}x}. \qquad (5.20)$$

Hence, the solution will be

$$u(t,x) = e_q^{\lambda t} e_q^{\sqrt{\lambda}x}. \qquad (5.21)$$

Let us consider now the symmetry reduction with respect to the operator B^q [15]. In this case introducing the appropriate symmetry variable $\eta = \frac{\beta_x x}{\sqrt{\beta_t t}}$ we get

$$u(x,t) = \frac{u_0}{\sqrt{\beta_t t}} \exp[-\frac{(\beta_x x)^2}{4\beta_t t}] 1. \qquad (5.22)$$

The solution (5.22) of the q–heat equation is meaningful as long as we are considering positives times and the value $t = 0$ is out of our time domain. In such a situation the solution (5.22) is entire and can be represented as a Taylor series and, thus, q–functions like the gaussian or the square root are meaningful. The method would not provide a meaningful q–function if we would consider all values of t. However, in $t = 0$ also the boost solution of the continuous heat equation would be singular and, thus, meaningless.

5.2   A generalized Hermite equation

Let us consider the following q–difference equation

$$\Delta_{xx}\psi + \beta_x x \Delta_x \psi = E\psi. \qquad (5.23)$$

By explicitating the operator β_x appearing in eq.(5.23) according to (3.11), we can rewrite it as

$$\Delta_{xx}\psi + x\psi_x = E\psi. \qquad (5.24)$$

By the q–umbral correspondence eq.(5.23) is related to the ordinary differential equation $\psi_{xx} + x\psi_x = E\psi$ whose solutions are given in terms of gaussian functions and Kummer confluent hypergeometric functions. Explicitly we have [21]:

$$\psi(x) = A_1 e^{-\frac{x^2}{2}} M(\frac{1}{2} + \frac{1}{2}E, \frac{1}{2}, \frac{1}{2}x^2) + A_2 x e^{-\frac{x^2}{2}} M(1 + \frac{1}{2}E, \frac{3}{2}, \frac{1}{2}x^2) \qquad (5.25)$$
$$= e^{-\frac{x^2}{2}}\phi(x).$$

Both the gaussian and the Kummer confluent hypergeometric functions have a power series expansions in terms of x,

$$\phi(x) = A_1[1 + (1 + \frac{1}{2}E)\frac{x^2}{2!} + (1 + \frac{1}{2}E)(3 + \frac{1}{2}E)\frac{x^4}{4!} + \ldots] + \qquad (5.26)$$
$$A_2[x + (2 + \frac{1}{2}E)\frac{x^3}{3!} + (2 + \frac{1}{2}E)(4 + \frac{1}{2}E)\frac{x^5}{5!} + \ldots].$$

So, the q–differential difference equation (5.24) will have a solution in terms of q–gaussian and q–Kummer confluent hypergeometric functions. Let us notice that, if $E = -2n$ with n a positive integer number, the q–Kummer confluent hypergeometric function is just a polynomial.

6   Conclusions

In this paper we present a q–extension of the umbral calculus and use it to provide solutions of linear q–difference and q–differential difference equations. In this way we obtain solutions which have a continuous limit.

The discretization procedure given by the recipe $\partial_x \longrightarrow \Delta_x$ and $x \longrightarrow \beta_x x$ works well for linear equations also in the case of q–shifts operators. In particular, it preserves the classical Lie symmetries which are described by linear equations.

We study in detail the behaviour of the q–exponential and q–gaussian functions and show their range of validity which depends from the q–discrete delta operator under consideration. The domain of convergence of the q–function to the continuous function is characterized in term of the zeroes of the q–function. The results are usually better in the case of symmetric q–delta operators.

Further work is in process on the complete description of a coherent q–umbral calculus and a comparison of the discrete and q–discrete solutions.

Acknowledgments

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academic Press, 1982, New York)

[2] L. Fadeev, Integrable Models in (1+1)-dimensional quantum field theory, in *Les Houches XXXIX Course*, eds.J.B. Zuber and R. Stora, (Elsevier, 1984, Amsterdam)

[3] *Non Extensive Thermodynamics and Physical Applications*, Eds. G. Kaniadakis, M. Lissia and A. Rapisarda, *Physica* **A305** Issue 1/2 (2002)

[4] D.S. MacAnally, *J. Math. Phys.* **36** (1995) 546

[5] D. Levi, J. Negro and M.A. del Olmo, *J. Phys. A: Math. Gen.* **34** (2001) 2023

[6] S. Roman and G.C. Rota, *Advanced Math.* **27** (1978) 95

[7] S. Roman, *The Umbral Calculus* (Academic, San Diego 1975)

[8] G.C. Rota, *Finite Operator Calculus* (Academic, San Diego 1975)

[9] D. Levi, P. Tempesta and P. Winternitz, Umbral calculus, difference equations and the discrete Schrödinger equation, arXiv:nlin.SI/0305047 (submitted to Comm. Math. Phys.)

[10] C Chryssomalakos and A Turbiner, *J. Phys. A: Math. Gen.* **34** (2001) 10475

[11] N. Aizawa, F.J. Herranz, J. Negro and M.A. del Olmo, *J. Phys. A: Math. Gen.* **35** (2002) 8179

[12] R. Floreanini and L. Vinet, *Lett. Math. Phys.* **32** (1994) 37

[13] R. Floreanini and L. Vinet, *J. Math. Phys.* **36** (1995) 3134

[14] D. Levi, L. Vinet and P. Winternitz, *J. Phys. A: Math. Gen.* **30** (1997) 633

[15] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York 1991)

[16] V. Kac and P. Cheung,*Quantum Calculus*, (Springer, New York, 2002)

[17] R. Bellman and K.L. Cooke, *Differential-difference equations*, (AP, New York, 1963)

[18] D. Levi, J. Negro and M.A. del Olmo, *Czechoslovak J. Phys.* **51** (2001) 341

[19] R. Floreanini, J. Negro, L.M. Nieto and L. Vinet, *Lett. Math. Phys.* **36** (1996) 351

[20] W. Miller, *Symmetry and Separation of Variables*, (Addison-Wesley, Reading (Ma), 1977)

[21] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965)